24/02/2021 MATH 2030B

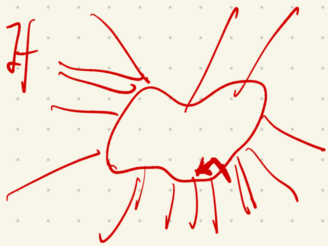
1. Cauchy Goursat Theorem.

i) Positive direction of the integral:

Counterclockwise: Positive

Clockwise: Negative.

If



Clockwise is positive.

Cauchy-Goursat Thm:

If a function f is analytic at all pt interior & on a simple closed contour C

then

$$\int_C f = 0.$$

Recall: simple closed means no self-intersection except at the 2 end pts of parametrization.

Pf: omitted.

2. Simply Connected Domain

Def: A domain such that $\forall$ closed simple contour only encloses pts in D, if we see the domain to be D.

Thm: If f is analytic throughout the simply connected domain D, then

$$\int_C f = 0$$

for $\forall$ closed contour C lying in D.

Remark: We do not need C to be simple.

Cor1: Remember the criterion for the existence of a cts function f:

existence of F $\Leftrightarrow$ $\forall$ closed contour in D,

$\int_C f = 0.$

For an analytic function on a simply connected domain, f always has its antiderivative.

Proof of the Thm:

i) The contour is simple: we can just apply Cauchy-Goursat Thm.

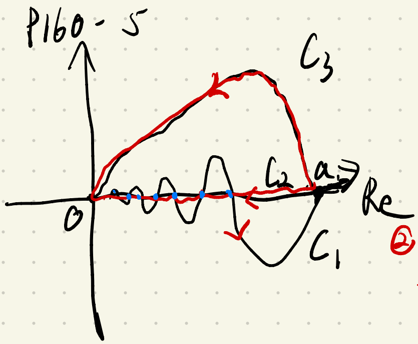
ii) The contour has self-intersection finite time:

Each part is simple, closed.

Then $\int_C f = \int_{C_1} f + \int_{C_2} f = 0.$

from (i).

iii) The contour has infinite self-intersections.



$\int_{C_1} f + \int_{C_3} f = 0$ ①

$-\int_{C_2} f + \int_{C_3} f = 0.$ ②

①-②

$\Rightarrow -\int_{C_2} f - \int_{C_1} f = 0$

denote $C = (C_1 \cup C_2)$

$\Rightarrow \int_C f = 0.$ $\square$

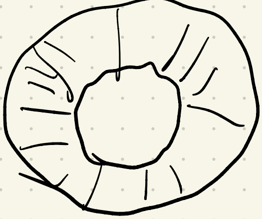
Cor 2: Entire function (Analytic on $\mathbb{C}$) have antiderivatives.

Pf: $\mathbb{C}$ is simply connected, By Cor 1, f has an antiderivative.

3. Multiply Connected Domain:

Def: Not simply connected domain:

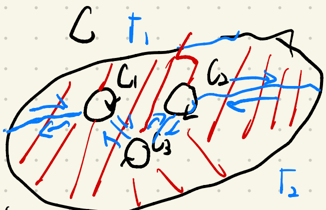
e.g



Thm: Suppose

i) C is simple closed contour, in positive direction.

ii) C_k ($k = 1, 2, \dots, n$) are simple closed contours interior to C, disjoint, their interiors have no pts in common.



Blue lines: L_1, L_2, L_3, L_4 (From L to R).

If f is analytic on these contours and on the region between these contours

$$\int_C f = -\sum_{k=1}^{n} \int_{C_k} f.$$

$$\int_{\Gamma_1 + L_1 + C_1' + L_2 + C_3' + L_3 + C_2' + L_4} f = 0 \qquad ①$$

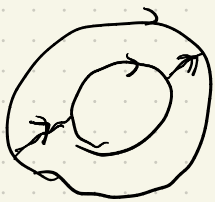
$$\int_{\Gamma_2 - L_4 + C_2'' - L_3 - C_3'' - L_2 - C_1'' - L_1} f = 0. \qquad ①$$

$① + ② \Rightarrow \int_{C + C_1 + C_2 + C_3} f = 0$

$\Rightarrow \int_C f = -\sum_{k=1}^{3} \int_{C_k} f$ Can use the method to show for $\forall n$. $\square$

Cor 3: If C_1 & C_2 are positive simple closed contours, C_1 interior to C_2. If f is analytic on & both interior to C_1 & C_2,

$$\int_{C_1} f = \int_{C_2} f .$$



One significance of Cor 3 is that we can deform some contour to some "good contour", e.g. circle.

4. Cauchy integral formula,

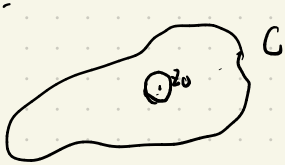
Thm: If f is analytic inside & on a simple closed contour in positive direction, & z_0 is $\forall$ pt inside C, then

$$f(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-z_0}\,dz .$$

$\Leftrightarrow$

$$f(s) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-s}\,dz, \quad \forall s \text{ interior to } C$$

Pf:



By Cor 3, we can take

$$C_e = \{ |z-z_0| = e \},$$

as $\frac{f(z)}{z-z_0}$ is analytic on & between C_e & C, $\int_C \frac{f(z)}{z-z_0} = \int_{C_e} \frac{f(z)}{z-z_0}$.

$$\int_C \frac{f(z)}{z-z_0} - \int_{C_e} \frac{f(z_0)}{z-z_0} = \int_{C_e} \frac{f(z)-f(z_0)}{z-z_0}$$

Also, $\int_{C_e} \frac{1}{z-z_0} = 2\pi i$

$$\int_C \frac{f}{z-z_0} - 2\pi i f(z_0) = \boxed{\int_{C_\rho} \frac{f - f(z_0)}{z-z_0}} \longrightarrow 0 \text{ as } |z-z_0| \to 0.$$

$= 0 \Rightarrow \int_C \frac{f}{z-z_0} = 2\pi i\, f(z_0)$.

f is analytic, $\Rightarrow f$ is continuous.

So for $\forall \varepsilon > 0$, $\exists \delta$ s.t. $|f(z) - f(z_0)| < \varepsilon$ whenever $|z-z_0| < \delta$.

Now take $\rho < \delta$.

$$\left| \int_{C_\rho} \frac{f - f(z_0)}{z-z_0} \right| \le \int_{C_\rho} \frac{|f - f(z_0)|}{|z-z_0|} \le 2\pi\rho \frac{1}{\rho} \cdot \varepsilon = 2\pi\varepsilon$$

And the ε is arbitrary, so

$$\int_{C_\rho} \frac{f - f(z_0)}{z-z_0} = 0. \qquad \square$$